Managing aggregation of shaped leaky buckets flows through GPS node in network calculus

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Abstract

The aim of this document is to solve some equations related to aggregate shaped leaky buckets flows with arrival curves $\bigwedge_{i\in[1,n]} \gamma_{r_i,b_i}$ and nodes with service curve $\beta_{R,T}$.

The two main results are the computation of the deconvolution $\bigwedge_{i\in[1,n]} \gamma_{r_i,b_i} \oslash \beta_{R,T}$, that allows to get the effect of the traversal of such a node by this kind of flow, and the computation of aggregate scheduling in FIFO node with such curves.

Disclaimer This report is about network calculus, as described in [LBT01]. There will be no network calculus recall here, and the reader should have a background in network calculus before reading this report.

Contents

1 Context

The leaky bucket is a very common model for specifying traffic constraint. In network calculus, such a constraint is modelled by a $\gamma_{r,b}$ arrival curve. Since the sum of two γ curves is still a γ curve, and the deconvolution of a γ curve by a $\beta_{R,T}$ curve is still a γ curve (up to subb-additive closure), a lot of studies have used this kind of traffic constraint. This stability by deconvolution is also true for a traffic constraint by two leaky buckets (arrival curve of the form $\gamma_{r,b} \wedge \gamma_{r',b'}$), another kind on constraint popular because this is one used in IntServ and named T-SPEC.

$$\gamma_{r,b}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ rt + c & \text{else} \end{cases} \tag{1}$$

Nevertheless, while studying the AFDX network [Gri04], another kind of curve have been discovered, that generalises the previous ones: $\bigwedge_{i \in [1,n]} \gamma_{r_i,b_i}$.

How does it comes from ? In AFDX network, the application traffic constraint can be modelled by a γ function, but we have shaping introduced by the medium (on a link with bandwidth D, the traffic is always constrained by λ_D, there can not be any burst greater than the link bandwidth), and the sum of functions of the form $\lambda_D \wedge \gamma_{r,b}$ is, in the general case, of the form $\bigwedge_{i \in [1,n]} \gamma_{r_i,b_i}$.

We could of course neglect the shaping, but experimentation has shown that is really decreases the computed bounds. This is why we decided to study this kind of arrival curves.

This report is organised as follow: section 2 gives some results useful in the proofs, section 3 defines a normal form for the functions $\bigwedge_{i \in [1,n]} \gamma_{r_i,b_i}$, and an algorithm to compute this normal form. Section 4 then gives the first main result; the effect of a $\beta_{R,t}$ server on such a flow. Section 5 handle the case when one such flow shares a FIFO server of service curve $\beta_{R,T}$ with a flow constrained by $\gamma_{r,b}$. Section 7 concludes.

Notations If X is a subset of $\mathbb{R}$, f, g two functions, $f =_X g$ means that $x \in X \Rightarrow f(x) = g(x)$. $\mathbb{R}_{\geq 0} \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid x \leq 0\}$ denotes the set of non negative reals.

2 Preliminary results

Theorem 1 (Sub-additive closure of $\gamma_{r,b} \oslash \beta_{R,T}$) *Assuming $R \geq r$:*

$$\overline{\gamma_{r,b} \oslash \beta_{R,T}} = \gamma_{r,b+rT}$$

Proof From [LBT01, §3.1.9], we have

$$\gamma_{r,b} \oslash \beta_{R,T} =_{\mathbb{R}_+} \gamma_{r,b+rT} \tag{2}$$

$$\gamma_{r,b} \oslash \beta_{R,T} \geq \gamma_{r,b+rT} \tag{3}$$

- First step: $\overline{\gamma_{r,b} \oslash \beta_{R,T}} \geq \gamma_{r,b+rT}$
  By isotonicity of sub-additive closure [LBT01, Theorem 3.1.11], eq. (3) $\Rightarrow \overline{\gamma_{r,b} \oslash \beta_{R,T}} \geq \overline{\gamma_{r,b+rT}}$ and $\overline{\gamma_{r,b+rT}} = \gamma_{r,b+rT}$ because γ functions are sub-additive.

- Second step: $\overline{\gamma_{r,b} \oslash \beta_{R,T}} \leq \gamma_{r,b+rT}$
  By definition $\overline{\gamma_{r,b} \oslash \beta_{R,T}} = \delta_0 \wedge (\gamma_{r,b} \oslash \beta_{R,T}) \wedge ((\gamma_{r,b} \oslash \beta_{R,T}) \otimes (\gamma_{r,b} \oslash \beta_{R,T})) \wedge \ldots$. It obviously follows, $\overline{\gamma_{r,b} \oslash \beta_{R,T}} \leq \delta_0 \wedge (\gamma_{r,b} \oslash \beta_{R,T})$.

  Because $\forall t \leq 0 : \delta_0(t) = 0$ we have $\forall t \leq 0 : \overline{\gamma_{r,b} \oslash \beta_{R,T}} = 0$. Combining this with equality of $\gamma_{r,b} \oslash \beta_{R,T}$ and $\gamma_{r,b+rT}$ on $\mathbb{R}_+$, we have $\delta_0 \wedge (\gamma_{r,b} \oslash \beta_{R,T}) = \gamma_{r,b+rT}$.

□

Corollary 1 (Sub-additive closure of $\gamma_{r,b} \oslash \delta_T$)

$$\overline{\gamma_{r,b} \oslash \delta_T} = \gamma_{r,b+rT}$$

PROOF Direct application of Theorem 1 with $\delta_T = \beta\infty, T$. □

3 Normal form of aggregation of shaped leaky buckets

Before giving the main results, and in order to simplify the proofs (and the implementations), we first define a normal form for the kind of function studied.

3.1 Definition of normal form and first properties

Definition 1 (Normal form of $\bigwedge_{i\in[1,n]} \gamma_{r_i,b_i}$) Let $\gamma_{r_1,b_1}, \ldots, \gamma_{r_n,b_n}$ be a set of γ functions. Let γ_i denotes γ_{r,i,b_i}.

The term $\bigwedge_{i\in[1,n]} \gamma_i$ is said to be *under normal form of minimum of γ functions*, iff

- there is no useless constraint:

$$\forall i, \exists t_i > 0, \forall j \neq i : \gamma_i(t_i) < \gamma_j(t_i) \tag{4}$$

- and the γ_i are sorted by decreasing rate

$$i < j \Rightarrow r_i > r_j \tag{5}$$

If $\bigwedge_{i\in[1,n]} \gamma_{r_i,b_i}$ is under normal form, the sequence $x_1, \cdots, x_{n+1}$ of intersection points, and $y_1, \cdots, y_{n+1}$ the intersection values, formally defined by:

$$\begin{cases} x_1 = 0 \\ \gamma_i(x_i) = \gamma_{i+1}(x_i) \quad \text{for } 1 \leq i \leq n \\ x_{n+1} = \infty \end{cases} \begin{cases} y_1 = b_1 \\ y_i = \gamma_i(x_i) \quad \text{for } 1 \leq i \leq n \\ x_{n+1} = \infty \end{cases} \tag{6}$$

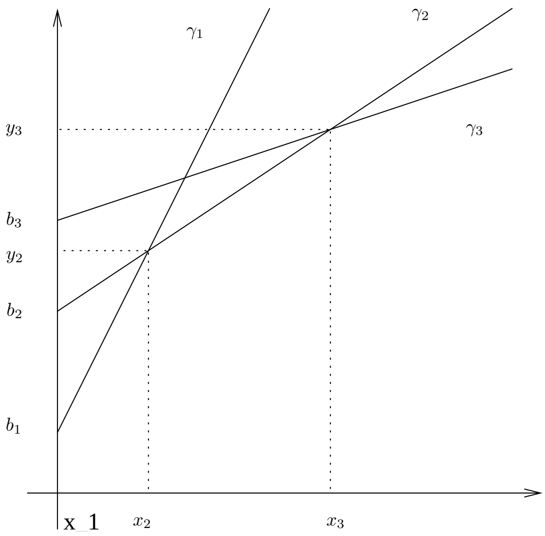

Figure 1: Set of γ functions under normal form

The condition (4) expresses that for all function γ_{r_i,b_i}, it exists (at least) one point t_i such that $\gamma_{r_i,b_i}(t_i) = \bigwedge_{i\in[1,n]} \gamma_{r_i,b_i}$, and, moreover, it is unique. An equivalent definition could have been that, if any i is removed, the function is not the same[1].

The condition (5) is just a usefull permutation.

The definition of the x_i is given to express the semantics of theses points. The value can of course be computed: $x_i = \frac{b_{i+1}-b_i}{r_i-r_{i+1}}$.

An example of such a set and related definitions is shown in figure 6.

Property 1 (Some properties of normal form of $\bigwedge_{i\in[1,n]} \gamma_{r_i,b_i}$)
If the expression $\bigwedge_{i\in[1,n]} \gamma_{r_i,b_i}$ is under normal form, as defined in definition 1, we have some properties:

- *the sequence x_i is increasing ($x_i < x_{i+1}$),*
- *the sequence b_i is increasing ($b_i < b_{i+1}$),*
- *the sequence r_i is decreasing ($r_i > r_{i+1}$),*
- *the function is piecewise linear*

$$\forall i \in [1,n] : \bigwedge_{j\in[1,n]} \gamma_j =_{[x_i,x_{i+1}]} \gamma_i$$

[1] $\forall i \in [1,n] : \bigwedge_{j\in[1,n],i\neq j} \gamma_{r_j,b_j} \neq \bigwedge_{j\in[1,n]} \gamma_{r_j,b_j}$

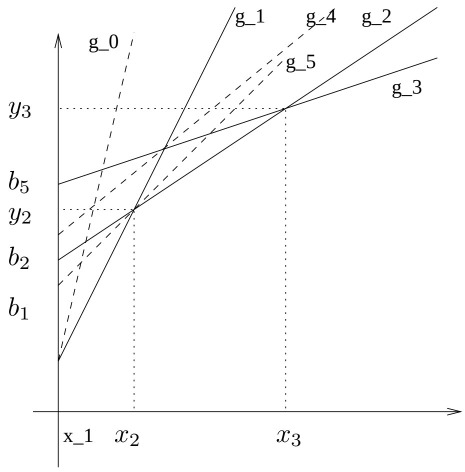

Figure 2: A $\bigwedge_i \gamma_i$ not under normal form

3.2 An algorithm to get normal form

Of course, in real life, the set of γ functions are not under normal form. Then, an effective way to get the normal form from a general set has to be build.

For example, we would like to be able to transform the set of $(\gamma_i)_{i\in[0,5]}$ of Figure 2 into the subset $(\gamma_i)_{i\in[1,3]}$ already seen at Figure 6.

The idea of the algorithm is the following. First, we have to find the γ_1 of normal form. This is the $\gamma_{r,b}$ function with the lesser b. If there are several function with the same b([2]), we chose the one with the lesser r. Then, we remove from the list all $\gamma_{r,b}$ with a rate r greater or equals to the one of γ_1. Then, to chose γ_2, the intersections points of γ_2 with all remaining γ_i is computed: γ_2 is the one with the "lesser" intersection point. Comptutation of γ_3 is done the same way, etc.

To simplify the proof, the algorithm assumes that the γ_{r_i,b_i} function are sorted on r_i in wide-sense decreasing order.

Theorem 2 (Algorithm 1 reduces a set of γ functions to is min normal form) *Applying the Algorithm 1 to a finite set of function $\{\gamma_i\}_{i\in[1,n]}$ such that $i<j \Rightarrow r_i \geq r_j$ returns a set $\{\gamma_i\}_{i\in I}$, with $I \subseteq [1,n]$, that satisfies the normal form conditions of Definition 1 (Equations (4) and (5)) such that:*

$$\bigwedge_{i\in[1,n]} \gamma_i = \bigwedge_{i\in I} \gamma_i$$

Notation For a variable x, notation $x^{|n}$ denotes the value of x just after execution of line n.

[2] like γ_0 and γ_x (TO BE FIX) in Figure 2

Algorithm 1 Reduction of a set of $\{\gamma_{r_i,b_i}\}_{i\in[1,n]}$ function to min normal form

Require: $n \geq 1$ and $\forall i, j \in [1, n] : i < j \Rightarrow r_i \geq r_j$

VAR
s : list of γ functions {Sorted but not under normal form}
nf : list of γ functions {Current normal form}
γ_n: a γ function {The new element to be added to nf}
γ_l: a γ function {The last element added to nf}
CODE
s $\leftarrow \{\gamma_i\}_{i\in[1,n]}$
$\gamma_n \leftarrow$ `findMinAtOrigin`(s)
s $\leftarrow \{\gamma_{r,b} \in s \mid r < r_n\}$
nf $\leftarrow (\gamma_n)$ {List of one element γ_n}
$\gamma_l \leftarrow \gamma_n$
while s $\neq \emptyset$ **do**
    $\gamma_n \leftarrow$ `minIntersection`(γ_l, s)
    nf $\leftarrow$ nf . (γ_n) { with . the concatenation operation on lists}
    s $\leftarrow \{\gamma_{r,b} \in s \mid r < r_n\}$
end while
return nf

with

`findMinAtOrigin` a function that inputs a set of γ_{r_i,b_i} function and return the one that have the minimum b_i and, if they are several, the one with the minimum r_i, and,

`minIntersection` a function that inputs a reference $\gamma_{r,b}$ function and a set of γ_{r_i,b_i} functions, and that computes the intersections points between $\gamma_{r,b}$ and each γ_{r_i,b_i} and return the one whith the minimum intersection abscisse; if they are several, it chooses the one with the lesser r_i.

Proof First step is that, by definition of `findMinAtOrigin`, just after execution of line 9, the min of s and $\{\gamma_n\}$ is the same function that $\{\gamma_i\}_{i\in[1,n]}$:

$$\gamma_n{}^{|9} \wedge \bigwedge_{\gamma_{r_i,b_i}\in s^{|9}} \gamma_{r_i,b_i} = \bigwedge_{i\in[1,n]} \gamma_i$$

Let denote $\{\gamma_{r_i,b_i}\}_{i\in J}$ the function removed from s. But, by construction, they all are greater than γ_n. So, they have no influence on the min. They can be safely removed.

Second is the loop: we have to determine a variant (to prove termination) and invariant (to prove the property). The variant is simple, this is the size of nf, that decreases at each iteration. The invariant states that:

- the min of elements of the union of s and nf is equals to the min of the initial set, and
- nf satisfies the normal form conditions of Definition 1, and
- the current γ_n is lesser than all other elements of nf from x_{n-1} up to ∞ (with the same definition of x_n that in Definition 1), and,
- all elements of s have an increase rate lesser than all elements of nf, and
- for all $t \in (0, x_{n-1}]$ the minimum is in nf, not in s: $\left(\bigwedge_{\gamma_i\in nf}\gamma_i\right)(t) < \left(\bigwedge_{\gamma_i\in s}\gamma_i\right)(t)$

Because at the end, nf is empty, then, s is the normal form of the min of the input set.

Each step of the loop preserves the invariant because: we know that γ_n is lesser than all elements of s at x_{n-1}, and that they all have an increase rate lesser than γ_n. Then, how long is γ_n lesser than the elements of s ? All intersections points are computed, and the one with the lesser x is the intersection with the first that becomes lesser than γ_n (notice that, since all functions are wide-sense increasing, the lesser x is also the lesser y). □

Complexity

Lemma 1 (Complexity of the Algorithm 1) *The Algorithm 1 has complexity* $\mathbf{O}(n^2)$

Proof Operation `findMinAtOrigin` (line 8) is at worst $\mathbf{O}(n)$. The same for `minIntersection` (line 13) and the loop (line 12) it executed at most n times.

We conjecture that there exist an $\mathbf{O}(n\log(n))$ algorithm, but in practise, we are handling problems of size about 10, and perhaps 100. In such little cases, looking for such an algorithm is useless.

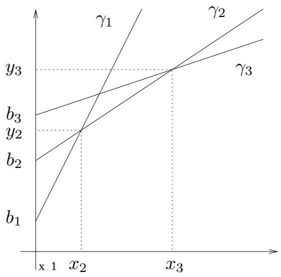

Figure 3: $\bigwedge_{i\in[1,3]} \gamma_i$ under normal form

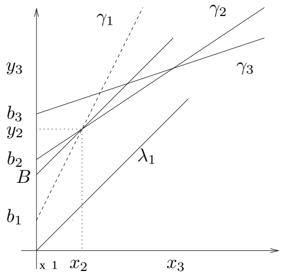

Figure 4: $\bigwedge_{i\in[1,3]} \gamma_i \oslash \lambda_1$

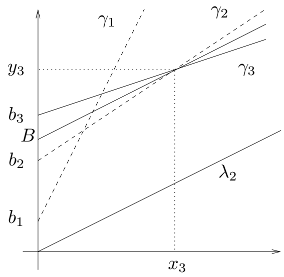

Figure 5: $\bigwedge_{i\in[1,3]} \gamma_i \oslash \lambda_{\frac{1}{2}}$

4 Deconvolution of normal form $\bigwedge \gamma_i$ by $\beta_{R,T}$

Here comes the first results on handling $\bigwedge \gamma_i$ flows: the deconvolution of such a flow by a $\beta_{R,T}$ function. The proof is done in two steps: firstly (Lemma 2), the deconvolution by λ_R is done, secondly (Lemma 3), the results is extended to $\beta_{R,T}$ by noticing that $\beta_{R,T} = \lambda_R \otimes \delta_T$ and $f \oslash (g \otimes h) = (f \oslash g) \oslash h$. Then, the main result comes: Theorem 3 compute the sub-additive closure of the previous result.

4.1 Deconvolution by λ_R

Lemma 2 (Deconvolution of normal form of $\bigwedge_i \gamma_i$ by λ_R) *Let $\bigwedge_{i\in[1..n]} \gamma_i$ be a function under normal form, as defined in Definition 1, and $\beta_{R,T}$ a rate-latency function such that $R \geq r_n$. Then, $\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \lambda_R$ can be computed on $\mathbb{R}_{\geq 0}$.*

$$\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \lambda_R =_{\mathbb{R}_+} \gamma_{R,B} \wedge \bigwedge_{i\in[k,n]} \gamma_i$$

$$\left(\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \lambda_R\right)(0) = B$$

with $k = \min \{i \mid r_i \leq R\}$, x_k beeing 0 if $k = 0$ and $B = y_k - R x_k$.

The graphical interpretation could be the following: the λ_R can be shifted up, as long as it intersects only with γ_i with higher slope. Figure 6 shows such a $\bigwedge_{i\in[1..n]} \gamma_i$ set, and its deconvolution by λ_1 in Figure 4 and by $\lambda_{\frac{1}{2}}$ in Figure 5.

Another point should be noticed: at 0, the deconvolution value is not nul.

Proof **Notations** In this proof, $\bigwedge_i \gamma_i$ is used as shorthand for $\bigwedge_{i\in[1,n]} \gamma_i$.

At first step, we can compute the deconvolution in one point $t > 0$ (the case $t = 0$ will be studied after).

$$\begin{aligned}\left(\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \lambda_R\right)(t) &= \sup_{0\leq s}\left\{\bigwedge_i \gamma_i(t+s) - \lambda_R(s)\right\}\\ &= \sup_{0\leq s}\left\{\bigwedge_i \gamma_i(t+s) - Rs\right\}\end{aligned}$$

Let be $f_t(s) = \bigwedge_i \gamma_i(t+s) - Rs$. Notice that $\gamma_{r_i,b_i}(t+s) - Rs = r_i(t+s) + b_i - Rs = (r_i - R)s + b_i + r_i t = r_i - R)(t+s) + b_i + Rt$. Then, $f_t(s) = \bigwedge_{i\in[1,n]} \gamma_{r_i-R,b_i}(t+s) + Rt$. It should be clear that, for j, $f_t =_{[x_j,x_{j+1}]} \gamma_{r_j-R,b_j} - Rt$, and $f'_t ==_{[x_j,x_{j+1}]} r_j - R$.

s		$x_k - t$	
f'_t	< 0	0	≥ 0
f_t	$\nearrow$		$\searrow$

The maximum of f_t can easily can be found: it is reached when $t + s = x_k$, and $\sup_{s\geq 0} f_t(s)$ depends of the ordering of x_k and t.

- if $t \leq x_k$, $\sup_{s\geq 0} f_t(s)$ is reached for $s = x_k - t$ and $\sup_{s\geq 0} f_t(s) = Rt + \bigwedge_{i\in[1,n]} \gamma_{r_i-R,b_i}(x_k) = Rt - Rx_k + \gamma_k(x_k)$
- if $t \geq x_k$, $\sup_{s\geq 0} f_t(s)$ is reached for $s = 0$ and $\sup_{s\geq 0} f_t(s) = \bigwedge_i \gamma_i(t)$.

If $t = 0$, we get the same result but the $\sup_{s\geq 0} f_t(s)$ is the limit to the left at 0, not a maximum. In this special case, we can have $k = 1$ *ie* $x_k = 0$. In this case, the limit to the left is not $Rt - Rx_k + \gamma_k(x_k)$ because γ_1 is not continuous at 0, neither $\bigwedge_i \gamma_i(0)$; but the value is b_1. This is why we have defined $y_1 = b_1$ and not $y_1 = \bigwedge_i \gamma_i(x_1) = 0$...

To sum up

$$\left.\begin{array}{l}\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \lambda_R =_{[0,x_k]} Rt + B\\ \left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \lambda_R =_{[x_k,\infty]} \bigwedge_i \gamma_i = \bigwedge_{i\in[k,n]} \gamma_i\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \lambda_R =_{\mathbb{R}_+} \gamma_{R,B} \wedge \bigwedge_{i\in[k,n]} \gamma_i\\ \left(\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \lambda_R\right)(0) = B\end{array}\right.$$

□

4.2 Deconvolution by $\beta_{R,T}$

Lemma 3 (Deconvolution of normal form of $\bigwedge_i \gamma_i$ by $\beta_{R,T}$) *Let $\bigwedge_{i\in[1..n]} \gamma_i$ be a function under normal form, as defined in Definition 1, and $\beta_{R,T}$ a rate-latency function such that $R \geq r_n$.*

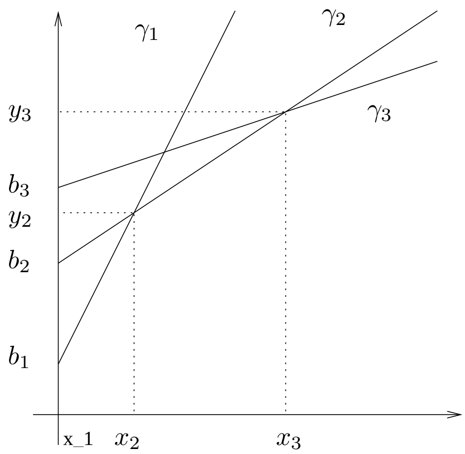

Figure 6: $\bigwedge_{i\in[1,3]} \gamma_i$

Then, $\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \beta_{R,T}$ *can be computed on* $\mathbb{R}_{\geq 0}$.

$$\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \beta_{R,T} =_{\mathbb{R}_+} \gamma_{R,B} \wedge \bigwedge_{i\in[k,n]} (\gamma_i \oslash \beta_{R,T})$$

$$\left(\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \beta_{R,T}\right)(0) = B \wedge \bigwedge_{i\in[k,n]} (b_i + r_i T)$$

with $B = RT + y_k - Rx_k$.

PROOF The proof is quite simple: just notice that $\beta_{R,T} = \delta_T \otimes \lambda_R$ and remind that $f \oslash (g \otimes h) = (f \oslash g) \oslash h$. Then, $\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \beta_{R,T} = \left(\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \lambda_R\right) \oslash \delta_T$. And, for continuous function, $(f \oslash \delta_T)(t) = f(t-T)$.

From previous Lemma, $\left(\bigwedge_{i\in[1..n]} \gamma_i\right) \oslash \beta_{R,T}(t) =_{\mathbb{R}_+} \gamma_{R,B} \wedge \bigwedge_{i\in[k,n]} \gamma_i$. And [LBT01, Figure 3.8] shows $(\gamma_i \oslash \beta_{R,T})(t) =_{\mathbb{R}_{\geq 0}} \gamma_i(t-T)$.

It should be noticed that, if all r_i are lesser or equal to R, ($k = 1$ then the term $\gamma_{R,B}$ is useless, because $\gamma_{R,B} = \gamma_{R,RT+b1} \geq \gamma_{r_1,b_1}$.

Moreover, if $T \geq x_k$, the term $\gamma_{R,B}$ is also useless: is comes from the fact that $\gamma_{R,B} \geq \gamma_{r_k,b'_k}$ (see Corollary 1 for details).

Corollary 1 (Properties of the $\boldsymbol{\gamma_{R,B}}$ term) *Here are a few properties of the terms of Lemma 3.*

(i) A big T *removes the* $\gamma_{R,B}$ *term: if* $T \geq x_k$*, the term* $\gamma_{R,B}$ *is useless.*

$$T \geq x_k \Rightarrow \gamma_{R,B} \wedge \bigwedge_{i\in[k,n]} (\gamma_i \oslash \beta_{R,T}) =_{\mathbb{R}_+} \bigwedge_{i\in[k,n]} (\gamma_i \oslash \beta_{R,T})$$

(ii) If $T \leq x_k$, functions $\gamma_{R,B}$ and $\bigwedge_{i\in[k,n]} (\gamma_i \oslash \beta_{R,T})$ intersect at point $x_k - T$.

$$\gamma_{R,B}(x_k - T) = y_k = \left(\bigwedge_{i\in[k,n]} (\gamma_i \oslash \beta_{R,T}) \right) (x_k - T)$$

PROOF Proof of (i) To prove that $\gamma_{R,B} \geq \gamma_{r_k,b'_k}$, we have to prove that $R \geq r_k$ (obvious, by definition of k) and $B \geq b'_k = b_k + r_k T$.

The definition of B is $B = y_k + R(T - x_k)$. So, under the assumption that $T \geq x_k$, we just have to prove that $y_k \geq b_k + r_k T$. It comes from $y_k = r_k x_k + b_k$. □

Proof of (ii)

$$\begin{aligned}\gamma_{R,B}(x_k - T) &= R(x_k - T) + B = Rx_k - RT + RT + y_k - Rx_k \\ &= y_k = \gamma_k(x_k) = \gamma_k(x_k + T - T) \\ &= (\gamma_k \oslash \beta_{R,T})(x_k - T)\end{aligned}$$

□

4.3 Sub-additive closure of the results

Theorem 3 (Sub-additive closure of deconvolution of normal form of $\bigwedge_i \gamma_i$ by $\beta_{R,T}$) .

Let $\{\gamma_i\}$ be a finite set of γ_{r_i,b_i} functions under the normal form of Lemma 3.

$$\begin{aligned}\overline{\left((\bigwedge_i \gamma_i) \oslash \beta_{R,T} \right)} &= \gamma_{R,B} \wedge \bigwedge_{r_i \leq R} (\overline{\gamma_{r_i,b_i} \oslash \beta_{R,T}}) \\ &= \gamma_{R,B} \wedge \bigwedge_{r_i \leq R} \gamma_{r_i,b_i + r_i T}\end{aligned}$$

with the same definition of B than in Lemma 3.

PROOF Going from the equality on $\mathbb{R}_+$ to equality of sub-additive closure is done the same way that for Theorem 1 and the fact that the convolution of star-shaped function nul at origin is simply the minimum of the functions [LBT01, Theorem 6.3.1].

We have:

$$\begin{aligned}&(\bigwedge_i \gamma_i) \oslash \beta_{R,T} \geq \gamma_{R,B} \wedge \bigwedge_{r_i \leq R} (\gamma_i \oslash \beta_{R,T}) \\ \Rightarrow\ &\overline{(\bigwedge_i \gamma_i) \oslash \beta_{R,T}} \geq \overline{\gamma_{R,B} \wedge \bigwedge_{r_i \leq R} (\gamma_i \oslash \beta_{R,T})}\end{aligned}$$

and the right term can be simplified

$$
\begin{aligned}
\overline{\gamma_{R,B} \wedge \bigwedge_{r_i \le R} (\gamma_i \oslash \beta_{R,T})} &= \overline{\gamma_{R,B}} \otimes \bigotimes_{r_i \le R} \overline{\gamma_i \oslash \beta_{R,T}} && \text{by [LBT01, Theorem 3.1.11]} \\
&= \gamma_{R,B} \otimes \bigotimes_{r_i \le R} \gamma_{r_i,b'_i} && \text{by Theorem 1} \\
&= \gamma_{R,B} \wedge \bigwedge_{r_i \le R} \gamma_{r_i,b'_i} && \text{by [LBT01, Theorem 6.3.1]}
\end{aligned}
$$

That is to say

$$
(\bigwedge_i \gamma_i) \oslash \beta_{R,T} \ge \gamma_{R,B} \otimes \bigotimes_{r_i \le R} \overline{\gamma_i \oslash \beta_{R,T}} = \gamma_{R,B} \wedge \bigwedge_{r_i \le R} \gamma_{r_i,b'_i}
$$

Moreover, by definition of sub-additive closure ($\overline{f} = \delta_0 \wedge f \wedge \ldots$), we always have $\overline{f} \le \delta_0 \wedge f$. And, for $f = (\bigwedge_i \gamma_i) \oslash \beta_{R,T}$ we have $\delta_0 \wedge f = \gamma_{R,B} \wedge \bigwedge_{r_i \le R} \gamma_{r_i,b'_i}$.

Both inequalities have been shown, then, equality follows. □

Corollary 2 (Sub-additive closure of deconvolution of normal form $\bigwedge_i \gamma_i$ by δ_T) *Let $\{\gamma_i\}$ a finite set of γ_{r_i,b_i} functions under the normal form of Lemma 3. Then*

$$
\overline{\left((\bigwedge_i \gamma_i) \oslash \delta_T \right)} = \bigwedge_i \left(\overline{\gamma_{r_i,b_i} \oslash \delta_T} \right)
$$

$$
\left((\bigwedge_i \gamma_i) \oslash \delta_T \right)(0) = \bigwedge_i \left(\gamma_{r_i,b_i + r_i T} \right)
$$

Proof The proof directory come from the observation that $\delta_T = \beta_{\infty,T}$. □

5 Aggregation of $\bigwedge \gamma_i$ sharing a FIFO β server

When several flows share a server, it is usefull to know which service is offered to each flow. This is called "aggregate scheduling" [LBT01, Chapter 6]. They are several results, depending of the server policy (blind, priority, FIFO).

If two flows R_1 and R_2 with arrival curves α_1 and α_2 share a server S with β service curve, we have some results depending on the server policy.

Blind the server offers to R_1 the service curve β_1 of (7), [LBT01, Theorem 6.2.1], under the conditions that β is a *strict* service curve for S, and that β_1 is a service curve

$$
\beta_1 = [\beta - \alpha_2]^+ \tag{7}
$$

Non-preemptive priority if the server is non-preemptive[3], and if R_1 has strictly higher priority, and if l_M^2 is the maximal size of a R_2 message,

[3] Which is often the case: the sending of a frame is never interrupted by the arrival of another one.

then [LBT01, Corollary 6.2.1], the server offers β_i to each flow R_i, as defined in (11), under the natural conditions that the β_i are service curves.

$$\beta_1 = [\beta - l_M^2]^+ \qquad\qquad \beta_2 = [\beta - \alpha_1]^+ \tag{8}$$

FIFO for each $\theta > 0$, the server offers to R_1 the service curve β_1^θ of (9), [LBT01, Proposition 6.2.1], under the natural condition that β_1^θ is a service curve

$$\beta_1^\theta(t) = [\beta(t) - \alpha_2(t-\theta)]^+ 1_{\{t>\theta\}} \tag{9}$$

with $1_{\{t>\theta\}} = 1$ if $t > \theta$, 0 otherwise. However, this does not gives one service curve, but a family. And it may not exist a θ better than all others. Nevertheless, we knows that the output flow R_1' is constrained by α_1' [LBT01, Proposition 6.2.2].

$$\alpha_1' = \inf_{\theta\geq 0} \left(\alpha_1 \oslash \beta_\theta^1\right) \tag{10}$$

The aim of this part is to study these results of aggregation of shaped leaky bucket.

First of all, it must be precised that we are not going the problem of two flows with arrival curves $\bigwedge_{i\in[1..n]} \gamma_{r_i,b_i}$ and $\bigwedge_{i\in[1..m]} \gamma_{r_i',b_i'}$ sharing a server, but only when one $\bigwedge_{i\in[1..n]} \gamma_{r_i,b_i}$ shares a server with a flow of arrival curve $\gamma_{r,b}$. This approximation is made because we are not able, up to now, to compute the general case, and because if a flow has $\bigwedge_{i\in[1..m]} \gamma_{r_i',b_i'}$ as arrival curve, it also have $\gamma_{r_m',b_m'}$ as arrival curve. Then, our results can also be applied in the general case, but it gives a pessimistic upper approximation.

That is to says, we only consider $\alpha_1 = \bigwedge_{i\in[1..n]} \gamma_{r_i,b_i}$, $\alpha_2 = \gamma_{r,b}$ and $\beta = \beta_{R,T}$.

Under the previous simplification, both blind and priority are very simple to compute, because the β_1 functions are of the form $\beta_{R,T}$ (11),(12).

$$[\beta_{R,T} - l_M]^+ = \beta_{R,T+\frac{l_M}{R}} \tag{11}$$

$$[\beta_{R,T} - \gamma_{r,b}]^+ = \beta_{R-r,\frac{RT+b}{R-r}} \tag{12}$$

Theorem 4 (Aggregation of $\bigwedge \gamma_i$ and $\gamma_{r,b}$ through a $\beta_{R,T}$ FIFO server) *Let $\{\gamma_1, \ldots, \gamma_n\}$ be a finite set of γ_{r_i,b_i} functions under the normal form of Lemma 3, let $\alpha_2 = \gamma_{r,b}$ and $\beta_{R,T}$ such that $R \geq r + r_n$. Then, we have*

$$\overline{\inf_{\theta\geq 0}\left(\bigwedge \gamma_i \oslash \beta_\theta^1\right)} \leq \gamma_{R',B'} \wedge \bigwedge_{r_i\leq R_i} \gamma_{r_i,b_i+r_i(T+\frac{b}{R})} \tag{13}$$

with $R' = R - r$, $B' = y_k + R'(T + \frac{b}{R} - x_k)$, $k = \min\{i \mid r_i \leq R'\}$ and $\beta_1^\theta(t) = [\beta(t) - \alpha_2(t-\theta)]^+ 1_{\{t>\theta\}}$

Proof The proof is given in appendix A. It is in french up to now.

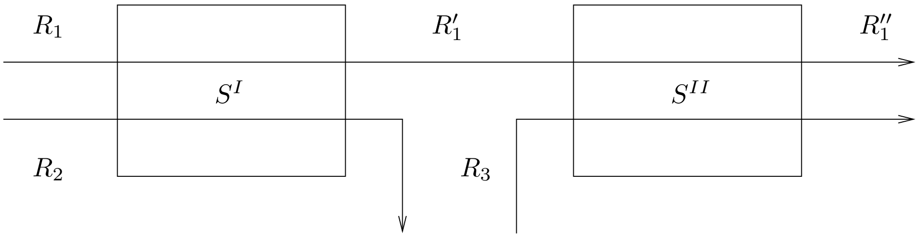

Figure 7: Three flows sharing two servers

The result is not very suprising: the formula really looks like the one of $\bigwedge \gamma_i \oslash \beta_{R,T}$ obtained in Theorem 3. $\gamma_{R',B'} \wedge \bigwedge_{r_i \leq R_i} \gamma_{r_i, b_i + r_i(T + \frac{b}{R})} = [\bigwedge \gamma_i \oslash \beta_{R', T+\frac{b}{R}}$

That is to say, a $\beta_{R,T}$ FIFO server is like a server with a lesser service rate $R' = R - r$, and a delay $T + \frac{b}{R}$, that is to say, the own delay of the server, T, plus eventually the time necessary to handle a burst form the other flow, $\frac{b}{R}$.

Moreover, it also means that $\theta = T + \frac{b}{R}$ is a good choice when trying to compute the service β_1^θ offered by the shared server to the flow R_1.

6 Example and comparison

To get an idea of the benefit of this method, we are going to study the example of Figure 7 with several methods.

Let assume that each flow R_i (resp. R_i') has α_i (resp. α_i') as an arrival curve, with $i \in \{1, 2, 3\}$ and each server S^k offers service curve $\beta^k = \beta_{R^k, T^k}$, with $k \in \{I, II\}$.

6.1 Curves are leaky bucket

In the first case, we are modeling the system with the well known leaky bucket constraint ($\alpha_i = \gamma_{r_i, b_i}$).

In this case, it is well known ([LBT01, Corollary 6.2.3]) that the optimal choice of θ for $\beta_1^\theta(t) = [\beta(t) - \alpha_2(t - \theta)]^+ 1_{\{t > \theta\}}$ (cf (9)) is $\theta = T^I + \frac{b_2}{R^I}$, and leads to $\alpha_1' = \gamma_{r_1, b_1 + r_1(T^I + \frac{b_2}{R^I})}$.

The delays d_2^I and d_3^{II} experimented by the flows R_2 and R_3 can be computed.

Lemma 4 (Delay of aggregation of two leaky bucket sharing a FIFO GPS node) *When two flows R_1, R_2 of respective arrival curve $\gamma_{r_1,b_1}, \gamma_{r_2,b_2}$ share a server with FIFO policy and service curve $\beta_{R,T}$ ($R \geq r_1 + r_2$), they both experiment the same maximum delay d defined by:*

$$d = T + \frac{b_1 + b_2}{R}$$

This result is easy to get. It is a bit surprising, because it means that the optimal choice of θ from delay point of view in equation 9 for each flow is $\theta = T + \frac{b_1+b_2}{R}$, as for tightening the output, the better choice is $\theta_1 = T + \frac{b_2}{R}$ for the flow R_1 and $\theta_2 = T + \frac{b_1}{R}$ for the flow R_2. But, it is not surprising from another point of view, since $T + \frac{b_1+b_2}{R}$ is the global delay for the aggregate flow $R_1 + R_2$, and in FIFO mode, the delay experimented by each flow can not be greater than the one of the global flow.

PROOF The proof is in appendix B.

From the previous Lemma 4, we have:

$$\begin{aligned} d_2^I &= T^I + \frac{b_1 + b_2}{R^I} \\ d_3^{II} &= T^{II} + \frac{b_1' + b_3}{R^{II}} \\ &= T^{II} + \frac{b_1 + b_3}{R^{II}} + \frac{r_1}{R^{II}} \left(T^I + \frac{b_2}{R^I} \right) \end{aligned}$$

To compute the end-to-end delay observed by the R_1, with the help of the "pay burst only once" principle, we have to compute the service offered by the concatenation of both servers S^I and S^{II}. Using the FIFO assumption and equation (9), for all θ^I and θ^{II}, $(\left[\beta^I(t) - \alpha_2(t - \theta^I)\right]^+ 1_{\{t>\theta\}}) \otimes (\left[\beta^{II}(t) - \alpha_2(t - \theta^{II})\right]^+ 1_{\{t>\theta\}})$. Using [LBT01, Corollary 6.2.3] once more, the best possible values are

$$\theta^I = T^I + \frac{b_2}{R^I} \qquad \qquad \theta^{II} = T^{II} + \frac{b_3}{R^{II}}$$

With these values

$$\begin{aligned} &(\left[\beta^I(t) - \alpha_2(t - \theta^I)\right]^+ 1_{\{t>\theta^I\}}) \otimes (\left[\beta^{II}(t) - \alpha_2(t - \theta^{II})\right]^+ 1_{\{t>\theta^{II}\}}) \\ =&\beta_{R^I - r_2, T^I + \frac{b_2}{R^I}} \otimes \beta_{R^{II} - r_3, T^{II} \frac{b_3}{R^{II}}} = \beta_{(R^I - r_2) \wedge (R^{II} - r_3), T^I + T^{II} + \frac{b_1}{R^I} + \frac{b_3}{R^{II}}} \end{aligned}$$

and the end-to-end delay d_2 can be computed:

$$d_2 = T^I + T^{II} + \frac{b_2}{R^I} + \frac{b_3}{R^{II}} + \frac{b_1}{(R^I - r_2) \wedge (R^{II} - r_3)}$$

It could be compared with the sum of the local delays d_1^I and d_1^{II}.

$$d_1^I + d_1^{II} = T^I + T^{II} + \frac{b_1 + b_2}{R^I} + \frac{b_1 + b_3}{R^{II}} + \frac{r_1}{R^{II}} \left(T^I + \frac{b_2}{R^I} \right)$$

6.2 Curves are shaped leaky buckets

In this section, the modeling of the arrival curves α_i is different. It is based on the observation that the arrival rate is constrained by the throughput D of the link. Then, the flows can be modelled by arrival curves of the form

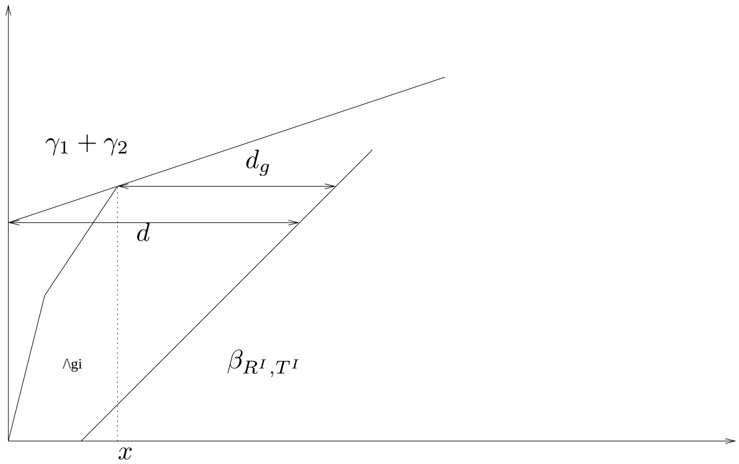

Figure 8: Benefit from the shaper modeling

$\alpha_i = \lambda_D \wedge \gamma_{r_i,b_i}$. In network calculus terminology, the link can be seen as a shaper with curve λ_D. The idea of adding the link constraint to the flow comes from [Gri04, GFF03].

The Figure 8 illustrates the benefits of the modelling of the shaper in your example: instead of computing the delay d, its gives d^g.

6.2.1 Computation with "distribution of aggregate delay"

But, since they were not able to compute convolution and deconvolution on such arrival curves, a conservative approximation was done, which can be called "distribution of aggregate delay". The idea is the following: even with this kind of curve, it is easy to compute the delay d experimented by the aggregate flow. And, in a FIFO server, the delay observed by each individual flow is at most the one of the aggregate flow. Then, the service offered to each individual flow can be approximated by a single delay δ_d.

Then, if d^I denotes the delay observed by the aggregate flow $R_1 + R_2$ in server S^I, this modelling give another expression of α'_1.

$$\alpha'_1 = \overline{(\lambda_D \wedge \gamma_{r_1,b_1}) \oslash \delta_{d^I}} = \gamma_{D,Dd^I} \wedge \gamma_{r_1,b_1+Dd^I}$$

We do not give an analytical expression of d^I, since it depends on the relations between r_1,b_1,r_2,b_2,D and R^I, but it can easily be computed by a simple ad-hoc tool.

Then, the delay d^{II} observed by the aggregate flow $R'_1 + R_3$ can be computed.

In this modelling, there is no way to compute the end-to-end delay, and d_3 is only the sum of the two local delays.

$$d_2 = d^I \qquad d_3 = d^{II} \qquad d_1 = d^I + d^{II}$$

Exp	R	T	r_1	b_1	r_2	b_2	r_3	b_3	d_1^o	d_1^g	d_1^b	d_2^o	d_2^g	d_2^b	d_3^o	d_3^g	d
E1	1	1	$\frac{1}{3}$	4	$\frac{1}{2}$	2	$\frac{1}{2}$	2	14	13.5	12	7	6	7	8	7.5	8
E2	1	1	$\frac{1}{2}$	2	$\frac{1}{3}$	4	$\frac{1}{3}$	4	13	14.33	12	7	6	9	9.5	8.33	1
E3	1	1	$\frac{1}{3}$	4	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{2}$	7.44	4.30	3.66	5.5	2.10	5.27	6.0	2.20	5
E4	1	1	$\frac{1}{3}$	4	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{2}$	7.44	4.30	3.66	5.5	2.10	5.27	6.0	2.20	5
E5	1	1	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{3}$	4	$\frac{1}{3}$	4	10.75	4.41	10.27	5.5	2.1	2.16	6.0	2.31	2
E6	10	1	$\frac{1}{3}$	4	$\frac{1}{2}$	2	$\frac{1}{2}$	2	2.82	2.44	2.42	1.60	1.22	1.41	1.64	1.22	1
E7	10	1	$\frac{1}{2}$	2	$\frac{1}{3}$	4	$\frac{1}{3}$	4	3.01	2.50	2.81	1.60	1.22	1.22	1.67	1.28	1
E8	1	$\frac{1}{4}$	$\frac{1}{3}$	4	$\frac{1}{2}$	2	$\frac{1}{2}$	2	12.5	11.81	10.5	6.25	5.25	6.25	7	6.56	7
E9	1	$\frac{1}{4}$	$\frac{1}{2}$	2	$\frac{1}{3}$	4	$\frac{1}{3}$	4	11.5	12.58	10.5	6.25	5.25	8.25	8.35	7.33	1

Table 1: Numerical comparison of the different strategies for the net of Figure 7

6.2.2 Computation with your results

With your new results, we can compute the service S_1^I and S_2^I offered by server S^I to the flow R_1 and R_2, using $T + \frac{b_j}{R}$ as a good choice for θ.

$$\beta_1^I = \beta_{R^I - r_2, T^I + \frac{b_2}{R^I}} \qquad \beta_2^I = \beta_{R^I - r_1, T^I + \frac{b_1}{R^I}}$$

With this curves and the tool presented in previous section, we are able to compute d_2 and d_1^I, the delays respectively observed by R_2 and R_1 in server S^I.

The same way, we can compute β_1^{II} and β_3^{II}. The expression of β_1^{II} is simple, but the one of β_3^{II} depend on the computation of $\alpha_1' = \overline{\inf_{\theta \geq 0} \left((\lambda_D \wedge \gamma_{r_1,b_1}) \oslash \beta_\theta^1 \right)}$, and as a complexe analytical expression.

$$\beta_1^{II} = \beta_{R^{II} - r_3, T^{II} + \frac{b_3}{R^{II}}}$$

6.2.3 Numerical examples

To compare the different strategies, we have defined some numerical configuration and computed the delays d_1, d_2 and d_3 of the flows R_1, R_2 and R_3. The results are presented in Table 1. In all the example, we have chosen $R^I = R^{II} = D$, and $T^I = T^{II}$. Notation d_i^x denotes the delay d of flow i computed with the method x, where $x = o$ is the original method, without shaping presented in Section 6.1, $x = g$ is the method of [Gri04, GFF03], and $x = b$ is our new method.

In experiments E1 and E2, the network elements are charged at 83%, and the size of the different flow are comparable. In experiments E3, E4 and E5, the network load is smaller (43%) and some big and small flows are mixed. Experiments E6 and E7 are the sames than E1 and E2, except that the network elements are 10 times more powerful, which lead to a small load (8%). At least, E8 and E9 are the sames than E1 and E2, but with smaller delays from the network elements.

The global conclusions of these experiments are: the method g is always better than o for local delays (d_2 and d_3), and in most cases better even for the end-to-end delay d_1 (all except E1). Your new method is also better than o for local delays (except for E9), and always better than o for end-to-end delay.

The comparison between g and b is not so clear: g is always better than b for local delays, and b is in general better than g for the end-to-end delay (always except E5).

The particularity of the E5 configuration is that the end-to-end flow is very small compared to the other. Then, the "pay burst only once" principle, that avoid to pay its own burst in each network element, gives a little gain compared to the cost of multiplexing with the other flows.

7 Conclusion

It had been shown in [Gri04] that taking into account the shaping of aggregate leaky bucket really decreases the bounds computed in network calculus.

This gives arrival curves of the form $\bigwedge \gamma_i$. We have calculated how to handle this kind of curve with $\beta_{R,T}$ service curves, event in case of aggregation.

We have done some experiments on some configuration that shows that our new formula are in general better than the older one.

We know have to test theses formulas on real case studies, to see which the benefits are in real configurations.

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A Proof of Theorem 4

At first step, let us consider the inner term $S_{\theta,t} \stackrel{\text{def}}{=} (\bigwedge \gamma_i \oslash \beta_\theta^1)(t)$ before trying to get its inf.

A.1 Solving $S_{\theta,t} = (\bigwedge \gamma_i \oslash \beta^1_\theta)(t)$

The definition of the convoluion is:

$$\left(\bigwedge \gamma_i \oslash \beta^1_\theta\right)(t) = \sup_{u\geq 0}\left\{\bigwedge \gamma_i(t+u) - \beta^1_\theta(u)\right\}$$

with

$$\beta^1_\theta(u) = [\beta_{R,T}(u) - \gamma_{r,b}(u-\theta)]^+ \, 1_{u>\theta}$$

Commençons par étudier $\beta^1_\theta(u) = [\beta_{R,T}(u) - \gamma_{r,b}(u-\theta)]^+ 1_{u>\theta}$.

A.1.1 Simplification du terme $\beta^1_\theta(u)$

$$\begin{aligned}
[\beta_{R,T}(u) - \gamma_{r,b}(u-\theta)]^+ &= (\beta_{R,T}(u) - \gamma_{r,b}(u-\theta)) \bigvee 0 \\
&= ((R(u-T)\bigvee 0) - \gamma_{r,b}(u-\theta)) \bigvee 0 \\
&= \begin{cases} R(u-T)\bigvee 0 & \text{si } u \leq \theta \\ ((R-r)u + r\theta - RT - b)\bigvee 0 & \text{sinon} \end{cases} \\
&= \begin{cases} \beta_{R,T}(u) & \text{si } u \leq \theta \\ (R'(u-T_\theta))\bigvee 0 & \text{si } u > \theta \end{cases} \\
&= \begin{cases} \beta_{R,T}(u) & \text{si } u \leq \theta \\ R'(u-T_\theta) & \text{si } u > T_\theta \text{ et } u > \theta \\ 0 & \text{si } u \leq T_\theta \text{ et } u > \theta \end{cases} \\
&\text{avec } R' = R - r \text{ et } T_\theta = \frac{RT + b - r\theta}{R - r}
\end{aligned}$$

Donc, lorsque l'on rajoute le terme $1_{\{u>\theta\}}$ on a:

$$\beta^1_\theta(u) = \begin{cases} 0 & \text{si } u \leq (\theta \vee T_\theta) \\ R'(u-T_\theta) & \text{si } u > (\theta \vee T_\theta) \end{cases} \tag{14}$$

It can also been expressed depending on the relation between θ and T_θ ($\theta \leq T_\theta \iff \theta \leq T + \frac{b}{R}$), removing u.

$$\begin{aligned}
\beta^1_\theta &= \beta_{R',T_\theta} && \text{if } \theta \leq T + \frac{b}{R}) \\
\beta^1_\theta &= \beta_{R',T_\theta} 1_{u>\theta} && \text{if } \theta \geq T + \frac{b}{R})
\end{aligned}$$

A.1.2 Report de l'expression de $\beta^1_\theta(u)$ dans $S_{\theta,t}$

Puisque $\beta^1_\theta(u)$ change d'expression en $\theta \vee T_\theta$, on peut décomposer la recherche du en deux partie: sur $[0, \theta \vee T_\theta]$ et sur $]\theta \vee T_\theta, \infty[$.

$$S_{\theta,t} = \sup_{0\le u\le \theta\vee T_\theta} \Big(\bigwedge \gamma_i(t+u)\Big)$$
$$\bigvee \sup_{u>\theta\vee T_\theta} \Big(\bigwedge \gamma_i(t+u) - R'(u-T_\theta)\Big)$$

Or, la fonction $\bigwedge \gamma_i$ étant croissante, le sup est atteint en $u = \theta \vee T_\theta$.

$$S_{\theta,t} = \underbrace{\bigwedge \gamma_i(t+\theta\vee T_\theta)}_{S^1_{\theta,t}} \bigvee \underbrace{\sup_{u>\theta\vee T_\theta} \Big\{\bigwedge \gamma_i(t+u) - R'(u-T_\theta)\Big\}}_{S^2_{\theta,t}} \tag{15}$$

A.1.3 Etude du terme $S^2_{\theta,t}$

Étudions de plus prêt le terme $S^2_{\theta,t}$. Pour cela, étudions la fonction $f_{t,\theta}$ définie par

$$f_{t,\theta} : \mathbb{R}_+ \to \mathbb{R}$$
$$u \mapsto \bigwedge \gamma_i(t+u) - R'(u-T_\theta) \tag{16}$$

On a bien

$$S^2_{\theta,t} = \sup_{u>\theta\vee T_\theta} \{f_{t,\theta}(u)\}$$

Etude de la fonction $f_{t,\theta}$ L'étude de la fonction $f_{t,\theta}$ est assez simple: elle a déjà été faite pour le calcul de $\bigwedge \gamma_i \oslash \beta_{R,T}$. La fonction est croissante "tant que" les pentes r_i sont supérieures à R', puis décroissent.

On rappelle les inégalités suivantes:

$$r_n < r_{n-1} < r_{n-2} < \cdots < r_1 \tag{17}$$
$$r_n \le R' \tag{18}$$

On note $k = \min\{i \mid r_i \le R'\}$, et x_k l'abscisse de l'intersection entre γ_{k-1} et γ_k, avec le cas particulier $x_1 = 0$. Alors, $f_{t,\theta}$ crois strictement jusqu'à $t+u = x_k$ puis décroît.

u		$x_k - t$	
$f'_{t,\theta}$	< 0	0	≥ 0
$f_{t,\theta}$	$\nearrow$		$\searrow$

Expression de $S^2_{t,\theta}$ D'après les variations présentées en (A.1.3), le calcul de $S^2_{t,\theta}$ est assez simple: soit $\theta \vee T_\theta$ est avant $x_k - t$, et le sup est atteint avec le max en $x_k - t$, soit $\theta \vee T_\theta \ge x_k - t$. Dans ce cas, $f_{t,\theta}$ décroît sur $\theta \vee T_\theta$ et son sup est la limite à gauche en ce point. Comme la fonction est continue, la limite à gauche est la valeur en ce point.

$$S^2_{t,\theta} = \begin{cases} f_{t,\theta}(x_k - t) & \text{si } x_k - t \geq \theta \vee T_\theta \\ f_{t,\theta}(\theta \vee T_\theta) & \text{si } x_k - t \leq \theta \vee T_\theta \end{cases} \tag{19}$$

A.1.4 Expression simplifiée de $S_{t,\theta}$

En reportant l'équation (19) dans (15), on obtient:

$$\begin{aligned} S_{\theta,t} &= \bigwedge \gamma_i(t + \theta \vee T_\theta) \bigvee \begin{cases} f_{t,\theta}(x_k - t) & \text{si } x_k - t \geq \theta \vee T_\theta \\ f_{t,\theta}(\theta \vee T_\theta) & \text{si } x_k - t \leq \theta \vee T_\theta \end{cases} \\ &= \bigwedge \gamma_i(t + \theta \vee T_\theta) \bigvee \begin{cases} \bigwedge \gamma_i(x_k) - R'(x_k - t - T_\theta) & \text{si } x_k - t \geq \theta \vee T_\theta \\ \bigwedge \gamma_i(t + \theta \vee T_\theta) - R'((\theta \vee T_\theta) - T_\theta) & \text{si } x_k - t \leq \theta \vee T_\theta \end{cases} \\ &= \begin{cases} (\bigwedge \gamma_i(t + \theta \vee T_\theta)) \vee (\bigwedge \gamma_i(x_k) + R'(t + T_\theta - x_k)) & \text{si } x_k - t \geq \theta \vee T_\theta \\ (\bigwedge \gamma_i(t + \theta \vee T_\theta)) \vee (\bigwedge \gamma_i(t + \theta \vee T_\theta) - R'((\theta \vee T_\theta) - T_\theta)) & \text{si } x_k - t \leq \theta \vee T_\theta \end{cases} \end{aligned}$$

Si on regarde de plus prêt le cas $x_k \leq \theta \vee T_\theta$, on voit que $R'((\theta \vee T_\theta) - T_\theta) \geq 0$, donc $\bigwedge \gamma_i(t + \theta \vee T_\theta) - R'((\theta \vee T_\theta) - T_\theta) \leq \bigwedge \gamma_i(t + \theta \vee T_\theta)$.

Donc, le deuxième cas pouvant se simplifier:

$$S_{\theta,t} = \begin{cases} (\bigwedge \gamma_i(t + \theta \vee T_\theta)) \vee (\bigwedge \gamma_i(x_k) + R'(t + T_\theta - x_k)) & \text{si } x_k - t \geq \theta \vee T_\theta \\ (\bigwedge \gamma_i(t + \theta \vee T_\theta)) & \text{si } x_k - t \leq \theta \vee T_\theta \end{cases}$$

S'il on distingue les relations d'ordre entre θ et T_θ, en qu'on substitue alors $\theta \vee T_\theta$ par son expression, on peut distinguer trois expressions, $S^a_{t,\theta}$, $S^b_{t,\theta}$ et $S^c_{t,\theta}$, définies par:

$$S_{\theta,t} = \begin{cases} S^a_{t,\theta} \stackrel{\text{def}}{=} (\bigwedge \gamma_i(t + \theta)) \ \vee (\bigwedge \gamma_i(x_k) + R'(t + T_\theta - x_k)) & \text{si } x_k - t \geq \theta \geq T_\theta \\ S^b_{t,\theta} \stackrel{\text{def}}{=} (\bigwedge \gamma_i(t + T_\theta)) \vee (\bigwedge \gamma_i(x_k) + R'(t + T_\theta - x_k)) & \text{si } x_k - t \geq T_\theta \geq \theta \\ S^a_{t,\theta} \stackrel{\text{def}}{=} (\bigwedge \gamma_i(t + \theta \vee T_\theta)) & \text{si } x_k - t \leq \theta \vee T_\theta \end{cases}$$

A.2 Calcul de $\inf_{\theta \geq 0} S_{t,\theta}$

La complexité du calcul de $\inf_{\theta \geq 0} S_{t,\theta}$ va venir du fait que $S_{t,\theta}$ change de forme en fonction des relations d'ordre entre $x_k - t$ (constant pour la recherche de $\inf_{\theta \geq 0}$), θ (que l'on fait varier), et T_θ, qui est une fonction de θ. On va donc devoir identifier des intervalles I sur lesquels la forme de $S_{t,\theta}$ est stable. Ensuite, une fois définit que $S_{t,\theta} = S^x_{t,\theta}$ pour $x \in \{a, b, c\}$ sur un intervalle I, il faudra calculer l'inf sur cet intervalle I, puis comparer les différents inf.

Nous allons dans un premier temps montrer comment les relations entre $x_k - t$, θ et T_θ se déduisent de relations entre $x_k - t$ et deux autres termes indépendants de θ (section A.2.1), puis faire un pré-calcul des inf des termes $S^a_{t,\theta}$, $S^b_{t,\theta}$ et $S^c_{t,\theta}$ (section A.2.2). Restera ensuite à intégrer tous les résultats dans tous les sous-cas (section A.2.3).

A.2.1 Relations entre $x_k - t$, θ et et T_θ

L'expression de $S_{t,\theta}$ changeant de forme en fonction des positions respectives de $x_k - t$, θ et T_θ, il nous faut pouvoir identifier cet ordre. De plus, comme nous allons beaucoup manipuler l'expression $x_k - t$, et que seul sa relation d'ordre avec deux autres variables x' et x'' sera utilisée, nous introduisons une notation plus compacte x^t.

$$x^t \stackrel{\text{def}}{=} x_k - t \tag{20}$$

$$\theta \geq T_\theta \iff \theta \geq x' \qquad x' \stackrel{\text{def}}{=} T + \frac{b}{R} \tag{21}$$

$$x^t \geq T_\theta \iff \theta \geq x'' \qquad x'' \stackrel{\text{def}}{=} \frac{RT + b + R'(t - x_k)}{r} \tag{22}$$

On remarque que x' et x'' sont indépendants de θ. On va donc pouvoir calculer $\inf_{\theta \geq 0} S_{t,\theta}$ en fonction des relations d'ordre entre x^t, x' et x''.

Tous les entrelacements entre x^t, x' et x'' peuvent-ils exister ? En fait, non, car $x'' \leq x' \iff x' \leq x^t$ (cf (23) et (24)).

$$x'' \leq x' \iff t \leq x_k - T - \frac{b}{R} \tag{23}$$

$$x' \leq x^t \iff t \leq x_k - T - \frac{b}{R} \tag{24}$$

If faut aussi noter que $T_\theta = \frac{RT+b-r\theta}{R-r}$ est une fonction décroissante de θ. Nous allons aussi avoir besoin de calculer T_θ pour certaines valeurs particulières de θ.

$$T_{x'} = x' \qquad\qquad T_{x''} = x^t$$

Proof (Expression de $T_{x'}$ et $T_{x''}$)

$$\begin{aligned}
T_{x'} &= \frac{RT + b - rx'}{R'} = \frac{1}{R'}\left(RT + b - r\left(T + \frac{b}{R}\right)\right) = \frac{1}{R'}\left(RT + b - rT - \frac{r}{R}b\right) \\
&= \frac{1}{R'}\left(R'T + \frac{R'}{R}b\right) = T + \frac{b}{R} = x' \\
T_{x''} &= \frac{RT + b - rx''}{R'} = \frac{1}{R'}\left(RT + b - r\frac{RT + b + R'(t - x_k)}{r}\right) \\
&= \frac{1}{R'}\left(RT + b - RT - b - R'(t - x_k)\right) = x_k - t
\end{aligned}$$

Proof (Preuves de (23) et (24)) • Preuve de (23)

$$\begin{aligned} x'' \leq x' &\iff RT + b + R'(t - x_k) \leq r(T + \frac{b}{R}) \\ &\iff R't \leq (r - R)T + \frac{r}{R}b - b + R'x_k \\ &\iff R't \leq -R'T - \frac{R'}{R}b + R'x_k \\ &\iff t \leq x_k - T - \frac{b}{R} \end{aligned}$$

- Preuve de (24)

$$x' \leq x^t \iff T + \frac{b}{R} \leq x_k - t \iff t \leq x_k - T - \frac{b}{R}$$

A.2.2 Etude des variations des termes $S^a_{t,\theta}$, $S^b_{t,\theta}$ et $S^c_{t,\theta}$

Nous pouvons faire une pré-étude des termes $S^a_{t,\theta}$, $S^b_{t,\theta}$ et $S^c_{t,\theta}$, pré-étude qui sera ensuite utile pour calculer l'inf de ces termes dans des intervalles particuliers.

Etude de $\inf_{m\leq\theta\leq M} S^a_{t,\theta}$ Ce terme est celui aux variations par rapport à θ les plus complexes. Rappelons sa forme

$$S^a_{t,\theta} = \Big(\bigwedge \gamma_i(t+\theta)\Big) \vee \Big(\bigwedge \gamma_i(x_k) + R'(t + T_\theta - x_k)\Big)$$

Il s'agit du max entre deux termes, $\bigwedge \gamma_i(t+\theta)$, qui est croissant par rapport à θ, et $\bigwedge \gamma_i(x_k) + R'(t + T_\theta - x_k)$ qui lui est décroissant (puisque T_θ est décroissant par rapport à θ).

Or, il n'existe pas de relations exacte entre $\inf(f \vee g)$ et les termes $\inf(f)$ et $\inf(g)$. On a juste l'encadrement donné dans l'équation 25, mais il peut être très grossier.

$$\sup(f) \vee \sup(g) \geq \inf(f \vee g) \geq \inf(f) \vee \inf(g) \tag{25}$$

En supposant que f est croissante et g décroissante, et que tous deux sont continues sur un intervalle $I = [m, M]$, on peut identifier plusieurs sous-cas: soit il n'y a pas d'intersection entre les deux courbes sur I (cas (i) et (i) de la Figure 9), et l'inf de $f \vee g$ est l'inf de f ou de g, que l'on peut facilement calculer ($f(m)$ ou $g(M)$), soit il existe un point d'intersection x, et $\inf_I(f \vee g) = f(x) = g(x)$. Si le calcul effectif est trop complexe dans notre cas, on peut tout de même obtenir une approximation.

$$f(M) \wedge g(m) \geq \inf_{[m,M]}(f \vee g) \geq f(m) \vee g(M) \tag{26}$$

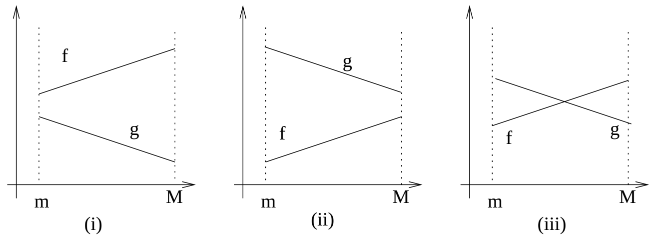

Figure 9: Les différents cas pour $\inf_I(f \vee g)$ avec $f \nearrow$ et $g \searrow$

Etude de $\inf_{m\leq\theta\leq M} S^b_{t,\theta}$

$$S^b_{\theta,t} = \Big(\bigwedge \gamma_i(t+T_\theta)\Big) \vee \Big(\bigwedge \gamma_i(x_k) + R'(t+T_\theta - x_k)\Big)$$

Ici, les deux termes $\bigwedge \gamma_i(t+T_\theta$ et $\bigwedge \gamma_i(x_k)+R'(t+T_\theta-x_k)$ sont décroissants par rapport à θ. Calculer l'inf de $S^b_{\theta,t}$, c'est donc calculer $\inf(f\vee g)$ avec f et g décroissantes. Dans ce cas, c'est assez simple:

$$\inf_{[m,M]}(f \vee g) = f(M) \vee g(M)$$

Mais on peut aller plus loin: on a donc

$$\inf_{[m,M]} S^b_{\theta,t} = \Big(\bigwedge \gamma_i(t+T_M)\Big) \vee \Big(\bigwedge \gamma_i(x_k) - R'(t+T_M - x_k)\Big)$$

Or, on peut calculer le plus grand de ces deux termes.

Lemma 1 *Soit u un réel positif*

$$\bigwedge \gamma_i(t+u) \leq \bigwedge \gamma_i(x_k) + R'(t-x_k+u)$$

PROOF

$$\begin{aligned}
& \bigwedge \gamma_i(t+u) \leq \bigwedge \gamma_i(x_k) + R'(t-x_k+u) \\
\iff\quad & \bigwedge \gamma_i(t+u) - R'u \leq \bigwedge \gamma_i(x_k) + R'(t-x_k) \\
\iff\quad & \bigwedge \gamma_i(t+u) - R'(u-T_\theta) \leq \bigwedge \gamma_i(x_k) - R'(x_k - t - T_\theta) \\
\iff\quad & f_{t,\theta}(u) \leq f_{t,\theta}(x_k - t)
\end{aligned}$$

On retrouve la fonction $f_{t,\theta}$ définie dans l'équation (16), et étudiée au paragraphe A.1.3, dont on sait qu'elle a sont maximum en $x_k - t$. $\square$

Donc, avec $u = T_M$, $\bigwedge \gamma_i(t+T_M) \leq \bigwedge \gamma_i(t+x_k) - R'(x_k - T_M)$.
En conclusion:

$$\inf_{[m,M]} S^b_{\theta,t} = \bigwedge \gamma_i(x_k) + R'(t+T_M - x_k) \tag{27}$$

Etude de $\inf_{m \leq \theta \leq M} S^c_{t,\theta}$ Le terme est très simple:

$$S^c_{t,\theta} = \bigwedge \gamma_i(t + \theta \vee T_\theta)$$

Si l'on se place sur un intervalle $[m, M]$ ou la relation entre θ et T_θ ne change pas (ce qui sera le cas dans la suite), seul un des deux terme reste dans l'expression. Comme $\bigwedge \gamma_i$ est une fonction croissante, et que T_θ est une fonction décroissante, on a:

$$\inf_{m \leq \theta \leq M} S^c_{t,\theta} = \begin{cases} \bigwedge \gamma_i(t + m) & \text{si } \theta \geq T_\theta \text{ sur } [m, M] \\ \bigwedge \gamma_i(t + T_M) & \text{si } \theta \leq T_\theta \text{ sur } [m, M] \end{cases} \tag{28}$$

A.2.3 Calcul de $S_{t,\theta}$ dans les différents cas

Nous allons maintenant étudier les 2 relations d'ordre possibles entre les trois constantes x^t (4), x' et x'', et en déduire une expression de $S_{t,\theta}$. Dans le cas où le calcul exact serait trop complexe (tous ces calculs sont fait pour être implantés), nous donnerons une sur-approximation. En effet tous ces calculs sont réalisés pour obtenir une courbe d'arrivée α à des flux R. Or, on sait que si α est une courbe d'arrivée de R, et si $\alpha' \geq \alpha$, α' est aussi une courbe d'arrivée de R, mais moins précise. Ici, si α est incalculable, nous nous contenterons de α'.

La démarche de calcul de $\inf_{\theta \geq 0} S_{t,\theta}$ consiste à décomposer $[0, \infty[$ en intervalles donnant la forme de $S_{t,\theta}$, de calculer l'inf sur chaque intervalle, puis de faire le minimum entre les différents termes. expliqué en détail, et les suivants plus rapidement.

Calcul de $S_{t,\theta}$ **si** $x'' \leq x' \leq x^t$ **(c-à-d** $t \leq x_k - T - \frac{b}{R}$**)**

θ		x''		x'		x^t	
$\theta \# T_\theta$	$\theta \leq T_\theta$			|	$\theta \geq T_\theta$		
$x^t \# T_\theta$	$x^t \leq T_\theta$	|	$x^t \geq T_\theta$				
$\theta \# x^t$	$\theta \leq x^t$					|	$\theta \geq x^t$
	(1)	|	(2)	|	(3)	|	(4)
$S_{t,\theta}$	$S^c_{t,\theta}$	|	$S^b_{t,\theta}$	|	$S^a_{t,\theta}$	|	$S^c_{t,\theta}$

Nous avons donc dans ce cas:

$$\inf_{\theta \geq 0} S_{t,\theta} = \inf_{0 \leq \theta \leq x''} S^c_{t,\theta} \wedge \inf_{x'' \leq \theta \leq x'} S^b_{t,\theta} \wedge \inf_{x' \leq \theta \leq x^t} S^a_{t,\theta} \wedge \inf_{x^t \leq \theta} S^c_{t,\theta}$$

Étude de chaque terme Étudions chaque terme:

- $\inf_{0 \leq \theta \leq x''} S^c_{t,\theta}$

$$\begin{aligned} \inf_{0 \leq \theta \leq x''} S^c_{t,\theta} &= \inf_{0 \leq \theta \leq x''} \left(\bigwedge \gamma_i(t + \theta \vee T_\theta) \right) \\ &= \bigwedge \gamma_i(t + T_{x''}) && \text{d'après (28)} \\ &= \bigwedge \gamma_i(x_k) && \text{car } T_{x''} = x_k - t \end{aligned}$$

[4] x^t est un raccourci pour $x_k - t$

- $\inf_{x''\leq\theta\leq x'} S^b_{t,\theta}$ Il suffit d'appliquer l'équation (27)

$$\inf_{x''\leq\theta\leq x'} S^b_{t,\theta} = \bigwedge \gamma_i(x_k) + R'(t + T'_x - x_k) \quad = \bigwedge \gamma_i(x_k) + R'(t + x' - x_k)$$

- $\inf_{x'\leq\theta\leq x_k} S^a_{t,\theta}$

  On est dans le cas $f(m) \leq g(m)$ et $f(M) \geq g(M)$, soit le cas le plus défavorable.

  En effet:

$$f(x') = \bigwedge \gamma_i(t + x') \qquad g(x') = \bigwedge \gamma_i(x_k) + R'(t + T_{x'} - x_k)$$
$$= \bigwedge \gamma_i(x_k) + R'(t - x_k + x')$$

  Or, d'après le Lemme 1, $\bigwedge \gamma_i(t + u) \leq \bigwedge \gamma_i(x_k) + R'(t - x_k + u)$, donc, avec $u = x'$, $f(x') \leq g(x')$.

$$f(x^t) = \bigwedge \gamma_i(x_k) \qquad g(x^t) = \bigwedge \gamma_i(x_k) + R'(t + T_{x^t} - x_k)$$

  Or, $t + T_{x^t} - x_k \leq 0$, car on sait que $T_u - u \leq 0 \iff u \geq T_u \iff u \geq x'$ (avec $u = x^t$). Donc, $f(x^t) \geq g(x^t)$.

  La seule conclusion que l'on puisse tirer est donc un encadrement

$$f(x^t) \wedge g(x') \quad \geq \quad \inf_{x'\leq\theta\leq x_k} S^a_{t,\theta} \quad \geq \quad f(x') \vee g(x^t)$$

  La borne supérieure se simplifie bien.

$$f(x^t) \wedge g(x') = \bigwedge \gamma_i(x_k) \wedge \big(\bigwedge \gamma_i(x_k) + R'\underbrace{(x^t - x')}_{\geq 0}\big) = \bigwedge \gamma_i(x_k)$$

  Par contre, la borne inférieure se simplifie mal.

$$f(x') \vee g(x^t) = \bigwedge \gamma_i(t + x') \vee \bigwedge \gamma_i(x_k) + R'(T_{x^t} - x^t)$$

  On a $\bigwedge \gamma_i(t + x') \leq \bigwedge \gamma_i(x_k)$ ([5]) et $R'(T_{x^t} - x^t) \leq 0$, donc pas de comparaison simple entre les deux termes.

  On ne peut pas non plus utiliser le Lemme 1, puisqu'avec $u = x^t$, on a une tautologie ($\bigwedge \gamma_i(x_k) \leq \bigwedge \gamma_i(x_k)$) et avec $u = T_{x^t}$, on obtient juste $\bigwedge \gamma_i(t + T_{x^t}) \leq g(x^t)$, mais $\bigwedge \gamma_i(t + T_{x^t}) \leq \bigwedge \gamma_i(x_k)$...

  En conclusion, on a, pour $x'' \leq x' \leq x^t$:

$$\bigwedge \gamma_i(x_k) \quad \geq \quad \inf_{x'\leq\theta\leq x_k} S^a_{t,\theta} \quad \geq \quad (\bigwedge \gamma_i(t + x')) \vee (\bigwedge \gamma_i(x_k) + R'(t + T_{x^t} - x_k))$$

- $\inf_{x^t\leq\theta} S^c_{t,\theta}$

$$\inf_{x^t\leq\theta} S^c_{t,\theta} = \inf_{x^t\leq\theta} \Big(\bigwedge \gamma_i(t + \theta \vee T_\theta)\Big) = \inf_{x^t\leq\theta} \Big(\bigwedge \gamma_i(t + \theta)\Big) = \bigwedge \gamma_i(x_k)$$

  On retrouve le premier terme.

[5] Puisque $x' \leq x^t$, d'où $t + x' \leq x_k$, et $\bigwedge \gamma_i$ croissante...

Calcul du min des termes Donc, en rassemblant les différents termes, on a, pour $x'' \leq x' \leq x^t$

$$\inf_{\theta \geq 0} S_{t,\theta} = \inf_{0 \leq \theta \leq x''} S^c_{t,\theta} \wedge \inf_{x'' \leq \theta \leq x'} S^b_{t,\theta} \wedge \inf_{x' \leq \theta \leq x^t} S^a_{t,\theta} \wedge \inf_{x^t \leq \theta} S^c_{t,\theta}$$

$$\inf_{\theta \geq 0} S_{t,\theta} = \bigwedge \gamma_i(x_k) \wedge \bigwedge \gamma_i(x_k) + R'(t + x' - x_k) \wedge \inf_{x' \leq \theta \leq x^t} S^a_{t,\theta} \wedge \bigwedge \gamma_i(x_k)$$

Or $x' \leq x^t$, donc $R'(x' - x^t) \leq 0$, donc $\bigwedge \gamma_i(x_k) + R'(t + x' - x_k) \leq \bigwedge \gamma_i(x_k)$

$$\inf_{\theta \geq 0} S_{t,\theta} = \bigwedge \gamma_i(x_k) + R'(t + x' - x_k) \wedge \inf_{x' \leq \theta \leq x^t} S^a_{t,\theta}$$

Comme nous n'avons qu'un encadrement du terme portant sur $S^a_{t,\theta}$, on ne peut faire qu'un encadrement du résultat:

$$S_M \geq \inf_{x' \leq \theta \leq x_k} S^a_{t,\theta} \geq S_m$$

avec

$$S_M \stackrel{\text{def}}{=} \left(\bigwedge \gamma_i(x_k) + R'(t + x' - x_k)\right) \wedge \left(\bigwedge \gamma_i(x_k)\right) = \bigwedge \gamma_i(x_k) + R'(t + x' - x_k)$$

$$S_m \stackrel{\text{def}}{=} \left(\bigwedge \gamma_i(x_k) + R'(t + x' - x_k)\right) \wedge \left(\bigwedge \gamma_i(t + x')\right) \vee \left(\bigwedge \gamma_i(x_k) + R'(t + T_{x^t} - x_k)\right)$$

On a $\bigwedge \gamma_i(t + x') \leq \bigwedge \gamma_i(x_k) + R'(t + x' - x_k)$ (Lemme 1, avec $u = x'$).

Mais comparer $\bigwedge \gamma_i(x_k) + R'(t + x' - x_k)$ et $\bigwedge \gamma_i(x_k) + R'(t + T_{x^t} - x_k)$ reviens à comparer x' et T^t_x. Pour comparer x' et T_{x^t}, il faut jouer avec leurs expressions respectives:

$$\begin{aligned} x' - T_{x^t} = T + \frac{b}{R} - \frac{RT + b - rx^t}{R'} &= \frac{1}{R - r}\left(RT + b - rT - \frac{r}{R}b - RT - b + rx^t\right) \\ &= \frac{r}{R'}\left(x^t - T - \frac{b}{R}\right) = \frac{r}{R'}(x^t - x') \end{aligned}$$

Comme nous sommes dans le cas $x' \leq x^t$, nous avons $x' \geq T_{x^t}$, soit $\bigwedge \gamma_i(x_k) + R'(t + x' - x_k) \geq \bigwedge \gamma_i(x_k) + R'(t + T_{x^t} - x_k)$. Ce qui nous donne que S_m est de la forme $S_m = a \wedge (b \vee c)$, avec $a \geq b$ et $a \geq c$, d'où l'inutilité du terme a.

En conclusion, pour $x'' \leq x' \leq x^t$, la meilleure approximation que nous puissions avoir est:

$$\bigwedge \gamma_i(x_k) + R'(t + x' - x_k) \geq \inf_{x' \leq \theta \leq x_k} S^a_{t,\theta} \geq \left(\bigwedge \gamma_i(t + x')\right) \vee \left(\bigwedge \gamma_i(x_k) + R'(t + T_{x^t} - x_k)\right)$$

On peut néanmoins simplifier un peu l'expression $R'(t + T_{x^t} - x_k)$

$$\begin{aligned} R'(t + T_{x^t} - x_k) = R'(T_{x^t} - x^t) &= RT + b - rx^t - R'x^t = RT + b - Rx^t \\ &= R(T + \frac{b}{R} - (x_k - t)) = R(t + x' - x_k) \end{aligned}$$

Ce qui nous amène à

$$\bigwedge \gamma_i(x_k)+R'(t+x'-x_k) \geq \inf_{x' \leq \theta \leq x_k} S^a_{t,\theta} \geq \Big(\bigwedge \gamma_i(t+x')\Big) \vee \Big(\bigwedge \gamma_i(x_k) + R(t+x'-x_k)\Big)$$

Calcul de $S_{t,\theta}$ si $x^t \leq x' \leq x''$ (c-à-d $t \geq x_k - T - \frac{b}{R}$)

θ		x^t	x'	x''
$\theta \# T_\theta$	$\theta \leq T_\theta$		$\theta \geq T_\theta$	
$x^t \# T_\theta$	$x^t \leq T_\theta$			$x^t \geq T_\theta$
$\theta \# x^t$	$\theta \leq x^t$	$\theta \geq x^t$		
	(1)	(2)	(3)	(4)
$S_{t,\theta}$	$S^c_{t,\theta}$	$S^c_{t,\theta}$	$S^c_{t,\theta}$	$S^c_{t,\theta}$

Dans ce cas, $S_{t,\theta}$ est toujours de la forme $S^c_{t,\theta}$, on peut donc décomposer l'analyse en fonction de $\theta \vee T_\theta$, sur $[0, x']$ et sur $[x', \infty)$, et on obtient, pour $x^t \leq x' \leq x''$:

$$\inf_{\theta \geq 0} S_{t,\theta} = \bigwedge \gamma_i(t+x')$$

A.3 Conclusion du calcul et première interprétation

Si on reporte systématiquement les résultats précédents, et qu'on remplace x', x'' et x^t par leurs expressions[6], on obtient:

- un simple encadrement si $t \leq x_k - T - \frac{b}{R}$ (c-à-d $x'' \leq x' \leq x^t$)

$$\bigwedge \gamma_i(x_k)+R'(t+x'-x_k) \geq \inf_{x' \leq \theta \leq x_k} S_{t,\theta} \geq \Big(\bigwedge \gamma_i(t+x')\Big) \vee \Big(\bigwedge \gamma_i(x_k) + R(t+x'-x_k)\Big) \tag{29}$$

- une valeur exacte pour $x^t \leq x' \leq x''$ (c-à-d $t \geq x_k - T - \frac{b}{R}$)

$$\inf_{\theta \geq 0} S_{t,\theta} = \bigwedge \gamma_i(t+x') \tag{30}$$

Mais ce qui nous intéresse est d'obtenir une fonction de t. Il nous faut donc faire réapparaître t que l'on avait caché lors de l'étude par rapport à θ.

Quelle interprétation avoir de tout cela ?

Le premier point à constater, c'est la prédominance du terme $\bigwedge \gamma_i(t+x')$. Que représente ce x' en terme de calcul réseau ? C'est en fait un simple décalage à gauche du flux d'entrée, de x', sans modification de la forme de la courbe d'arrivée α_1. Et que représente $x' = T + \frac{b}{R}$? Et bien, T est le délai propre du service β, et $\frac{b}{R}$ représente le temps nécessaire au serveur de service β pour traiter une rafale b du flux α_2.

[6] Ces notations n'ont été introduites que pour alléger les calculs lors de la recherche de l'inf par rapport à θ.

Donc, quand le flux de sortie à pour courbe de service $\bigwedge \gamma_i(t+x') = \bigwedge \gamma_i(t+T+\frac{b}{R}) = \bigwedge \gamma_i(t+T) \oslash \delta_{\frac{b}{R}}$, cela signifie simplement que le flux R_2 expérimente une simple retard $\frac{b}{R}$, nécessaire pour traiter une rafale de R_1, mais qu'une fois cela fait, il a "son tour dans la file" et plus de contrainte spécifique.

Cela correspond bien à ce que l'on attend d'un systèmes partagé FIFO: après que le serveur en a fini avec l'autre, c'est comme si on était seul.

On voit aussi que c'est le pire cas, c'est à dire que quand la solution n'est pas de cette forme là, c'est que ce terme a disparu au cours des calculs, et que la solution est plus petite que ce terme.

Le terme $\bigwedge \gamma_i(x_k) + R'(t+x'-x_k)$ ressemble lui au terme $\gamma_{R,B'}$ qui apparaît dans le calcul de la déconvolution d'un terme $\bigwedge \gamma_i$ par une fonction $\beta_{R,T}$.

On note aussi que la solution calculée est continue: au point $t = x_k - x'$, on a $\bigwedge \gamma_i(t+x') = \bigwedge \gamma_i(x_k)$, $\bigwedge \gamma_i(x_k) + R'(t+x'-x_k) = \bigwedge \gamma_i(x_k)$ et $\bigwedge \gamma_i(x_k) + R(t+x'-x_k) = \bigwedge \gamma_i(x_k)$.

Dans la mise en pratique, on ne va considérer en fait que la borne supérieure du résultat, car on sait que si α est la courbe d'arrivée d'un flux, et si $\alpha' \geq \alpha$, alors α' est aussi une courbe d'arrivée du flux. On a pas de résulta similaire sur une borne inférieure.

Si on cherche une expression unifié compacte, on peut noter que que sur $[x'-x_k]$, $\bigwedge \gamma_i(t+x') = \bigwedge_{i\geq k} \bigwedge \gamma_i(t+x') = \bigwedge_{i\geq k} \bigwedge \gamma_i(t) \oslash \delta_{x'} = \bigwedge_{r_i \leq R'} \bigwedge \gamma_i(t) \oslash \delta_{x'}$.

On peut aussi noter que pour $t > 0$, $\bigwedge \gamma_i(x_k) + R'(t+x'-x_k) = \gamma_{R',B'}$ avec $B' = \bigwedge \gamma_i(x_k) + R'(x'-x_k)$.

Comme, pour tout $t \leq x'-x_k$, $\gamma_{R',B'}(t) \leq \bigwedge_{r_i \leq R'} \bigwedge \gamma_i(t) \oslash \delta_{x'}$, on a

$$\forall t > 0 : \inf_{\theta \geq 0} \left(\bigwedge \gamma_i \oslash \beta^1_\theta \right) \leq (\gamma_{R',B'} \wedge \bigwedge_{r_i \leq R'} \gamma \oslash \delta_{T+\frac{b}{R}})(t) \tag{31}$$

Il reste à régler le cas de la valeur en 0. Il faut pour cela passer à la clôture sous-additive.

Donc, on peut dire que si un flux R de courbe d'arrivée $\bigwedge \gamma_i$ est agrégé avec un autre flux de courbe d'arrivée $\alpha_{r,b}$ en entrée d'un serveur de courbe de service $\beta_{R,T}$, alors la fonction (32) est une courbe de service du flux de sortie R'.

$$\overline{\inf_{\theta \geq 0} \left(\bigwedge \gamma_i \oslash \beta^1_\theta \right)} \leq \gamma_{R',B'} \wedge \bigwedge_{r_i \leq R'} \overline{\gamma_i \oslash \delta_{T+\frac{b}{R}}} \tag{32}$$

avec $R' = R - r$, $B' = y_k + R'(T + \frac{b}{R} - x_k)$, et $k = \min\{i \,|\, r_i \leq R'\}$, et x_k l'abscisse de l'intersection entre γ_{k-1} et γ_k, $y_k = \gamma_i(x_k)$ avec le cas particulier $x_1 = 0$, $y_1 = b_1$.

B Proof of Lemma 4

The proof is done for flow R_1. For each θ, $\beta^1_\theta = [\beta(t) - \alpha_2(t-\theta)]^+ 1_{\{t>\theta\}}$, is a service curve of the service offered to R_1. It is shown in the proof of Theorem 4 (appendix A), that $\beta^1_\theta = \beta_{R-r_2,T_{1,\theta}} 1_{t>\theta}$, with $T_{1,\theta} = \frac{RT+b_2-r_2\theta}{R-r_2}$.

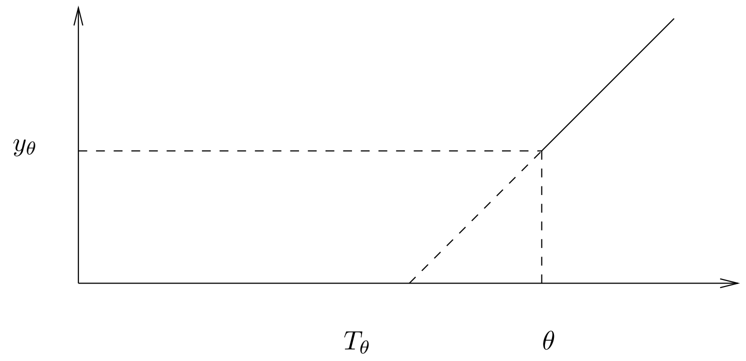

Figure 10: Function β_θ^1 for $\theta > T + \frac{b_2}{R}$

If $\theta \leq T + \frac{b_2}{R}$, $\beta_\theta^1 = \beta_{R-r_2,T_{1,\theta}}$, then, the delay $d_{1,\theta}$ can be computed:

$$d_{1,\theta} = T_{1,\theta} + \frac{b_1}{R - r_2} = \frac{RT + b_1 + b_2 - r_2\theta}{R - r_2}$$

In this case, the minimal delay for $0 \leq \theta \leq T + \frac{b_2}{R}$ is when $\theta = T + \frac{b_2}{R}$.

$$\begin{aligned}\inf_{0 \leq \theta \leq T + \frac{b_2}{R}} d_{1,\theta} &= \frac{1}{R - r_2}\left(RT + b_1 + b_2 - r_2(T + \frac{b_2}{R})\right) \\ &= T + \frac{b_1}{R - r_2} + \frac{b_2}{R}\end{aligned}$$

When $\theta > T + \frac{b_2}{R}$, the term $1_{t>\theta}$ can no more be neglected, and the service curve looks like the example of Figure 10. Then, let us use a graphical reasoning. If $b_1 \geq y_\theta$, the expression of $d_{1,\theta}$ is the same. And because $b_1 \geq y_\theta \iff b_1 \geq R\theta - RT - b_2 \iff \theta \geq T + \frac{b_1+b_2}{R}$. In this case, the minimum is reached when $\theta \geq T + \frac{b_1+b_2}{R}$. That is to say:

$$\begin{aligned}\inf_{T + \frac{b_2}{R} \leq \theta \leq T + \frac{b_1+b_2}{R}} d_{1,\theta} &= \frac{1}{R - r_2}\left(RT + b_1 + b_2 - r_2(T + \frac{b_1 + b_2}{R})\right) \\ &= T + \frac{b_1 + b_2}{R}\end{aligned}$$

In the other case, when $b_1 \leq y_\theta$, then $d_{1,\theta} = \theta$, and the minimum is reached when $\theta = T + \frac{b_1+b_2}{R}$.